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On the Relaxation Time of Gauss' Continued-Fraction Map. II. The Banach Space Approach (Transfer Operator Method)

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The spectrum of the transfer operator $\mathscr{L}$ for the map $Tx = 1/x - [1/x]$ when restricted to a certain Banach space of holomorphic functions is shown to coincide with the spectrum of the adjoint U^* of Koopman's isometric operator $Uf(x) = f \circ T(x)$ when the former is restricted to the Hilbert space $\mathscr{H}^2(v)$ introduced in part I of this work. If $\mathscr{N}$ denotes the operator $\mathscr{L} - P_1$ with P_1 the projector onto the eigenfunction to the dominant eigenvalue $\lambda_1 = 1$ of $\mathscr{L}$, then $-\mathscr{N}$ is a u_0-positive operator with respect to some cone and therefore has a dominant positive, simple eigenvalue $-\lambda_2$. A minimax principle holds giving rigorous upper and lower bounds both for λ_2 and the relaxation time of the map T.

KEY WORDS: Transfer operator; continued fraction; relaxation time; minimax principle; trace formulas.

1. INTRODUCTION AND RESULTS

In the first part of this paper,[1] denoted henceforth I, we discussed the Hilbert space approach for the relaxation time τ of Gauss' continued-fraction map $Tx = 1/x - [1/x]$, $x \neq 0$. The main object of study was the adjoint operator U^* of Koopman's operator $Uf(x) = f \circ T(x)$ in the Hilbert space $L_2(\mu)$, where μ denotes the Gauss measure $d\mu(x) = [1/(x+1)]\, dx$. If $\hat{K}$ denotes the restriction of U^* to the Hilbert space $\mathscr{H}^2(v)$, some Hardy space of holomorphic functions in the half-plane, then the spectrum of $\hat{K}$ turns out to be real and discrete: $\hat{K}$ is isomorphic to a generalized Hankel transform in some L_2 space over the positive real axis. Using in part

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numerical arguments, we showed the subdominant eigenvalue λ_2 of $\hat{K}$ to be simple and negative. Generalized Temple inequalities then allowed a rather accurate determination of λ_2 and therefore of τ through the relation $\tau = -1/\log|\lambda_2|$. Furthermore, interesting trace formulas for the operators $\hat{K}^n$ where proved for $n=1$ and $n=2$ relating these traces to the fixed points of the map T^n and conjectured for arbitrary n.

In this part of our investigation we discuss the Banach space approach; more precisely, the transfer or Perron–Frobenius operator $\mathscr{L}: L_1(dx) \to L_1(dx)$ of the map T defined quite generally as

$$\mathscr{L}f(x) = \sum_{y \in T^{-1}(x)} \frac{1}{|T'(y)|} f(y) \tag{1}$$

$\mathscr{L}$ describes just the transformation of densities under the map T. This operator is known to be closely related to the operator U^* and can therefore be used as well as the latter to characterize ergodic properties of the map T.[2] Some of the spectral properties of $\mathscr{L}$ for the Gauss map have been investigated[3,4] in relation to analyticity properties of generalized zeta functions for this system. They have been used quite recently by Pollicott[5] in his work on the distribution of periodic orbits of the geodesic flow on the modular surface. The main result proved in Refs. 3 and 4 is that $\mathscr{L}$ becomes a nuclear operator when restricted to a certain Banach space $A_\infty(D)$ of holomorphic functions over the disk $D = \{z \in \mathbb{C}: |z-1| < 3/2\}$, which, furthermore, is u_0-positive[11] with respect to a cone of positive functions in this space. In addition, simple trace formulas have been derived relating the trace of $\mathscr{L}^n$ to the fixed points of T^n for arbitrary n.

What we intend to do in the present paper is to clarify the connection between the above results on $\mathscr{L}$ and those on the operator $\hat{K}$ derived in I.

In fact, what we can show is the following: the spectrum $\sigma(\mathscr{L})$ of the operator $\mathscr{L}$ in the space $A_\infty(D)$ is identical to the spectrum $\sigma(\hat{K})$ of $\hat{K}$ in the space $\mathscr{H}^2(v)$. Therefore the trace formulas for $\mathscr{L}$ mentioned above are also valid for the operator $\hat{K}$. They lead to new identities for multiple integrals over products of Bessel functions. As a byproduct of this spectral identity, we get the result that $\sigma(\mathscr{L})$ is real, for which we do not know a direct proof. This, on the other hand, has interesting implications on the location of the poles in the zeta function of Ruelle for this system.[3]

In contrast to the discussion in I, we can prove in the Banach space setting without relying on numerical work that $\mathscr{L}$ has a subdominant eigenvalue λ_2, which is negative and simple. This follows from the fact that the operator $-\mathscr{N}$ in the decomposition $\mathscr{L} = P_1 + \mathscr{N}$ is also u_0-positive with respect to some cone C in a certain Banach space of holomorphic functions. This implies that $\mathscr{N}$ also can be decomposed as $\mathscr{N} = \lambda_2 P_2 + \mathscr{M}$, where λ_2 is its dominant eigenvalue in absolute value and P_2 is the projector onto the corresponding eigenfunction h_2. The $\mathscr{M}$ is a linear operator with $\mathscr{M}P_2 = P_2\mathscr{M} = 0$ and spectral radius strictly smaller than $-\lambda_2$. We expect $\mathscr{M}$ also to be u_0-positive with respect to some cone, and so on, so that finally all eigenvalues λ_i of $\mathscr{L}$ would be simple.

The property of $-\mathscr{N}$ to be u_0-positive ensures a minimax principle for λ_2, which at least in principle allows its exact determination. Anyhow, it gives rigorous upper and lower bounds for λ_2 and therefore also for the relaxation time τ of T, which are indeed rather good even for very simple test functions, as will be shown.

The paper is organized as follows: In the next section we introduce the transfer operator $\mathscr{L}$ for T, the relevant Banach space $A_{1,\infty}(D)$ of holomorphic functions, and recall some of the properties of $\mathscr{L}$ in this space, which follow from similar results in Refs. 3 and 4, where a slightly different space was used. In the next section we prove that $\sigma(\mathscr{L})$ in the space $A_{1,\infty}(D)$ is identical to $\sigma(\hat{K})$ in the space $\mathscr{H}^2(v)$ as discussed in I. We next define the cone C and show u_0-positivity of the operator $-\mathscr{N}$ with respect to this cone. From this a minimax principle for the eigenvalue λ_2 follows, giving upper and lower bounds. We close with two conjectures about a possible extension of these spectral results to more general composition operators in spaces of holomorphic functions arising, for instance, as transfer operators of locally expanding dynamical systems.

Some of our results have been obtained by Wirsing[6] in a somewhat different approach.

2. THE TRANSFER OPERATOR FOR T

The Perron–Frobenius, or as we prefer to call it, the transfer operator $\mathscr{L}$ for a general map T of the unit interval I is defined as the following linear bounded operator $\mathscr{L}\colon L_1(dx) \to L_1(dx)$ on the space of Lebesgue-integrable functions over I:

$$\mathscr{L}f(x) = \sum_{y \in T^{-1}x} \frac{1}{|T'(y)|} f(y)$$

where $T^{-1}x$ denotes the set of preimages of x. If f is a density, then $\mathscr{L}f$ is just the density after the action of T on I. In the case of the map $T(x) = 1/x - [1/x]$, $x \neq 0$, this gives

$$\mathscr{L}f(x) = \sum_{n=1}^{\infty} \frac{1}{(x+n)^2} f\left(\frac{1}{x+n}\right) \tag{2.1}$$

This is a special case of a class of transfer operators for T discussed in Refs. 3 and 4. There it was shown that $\mathscr{L}$ defines a nuclear operator of order zero[7],[2] when restricted to the Banach space $A_\infty(D)$ of holomorphic functions over the disk D with $D=\{z\in\mathbb{C}: |z-1|<3/2\}$. Another possible choice is the disk D_1 defined as $D_1=\{z\in\mathbb{C}: |z-1|<1\}$, which was used by Pollicott[5] in his recent work on the distribution of periodic orbits of the geodesic flow on the modular surface. In Ref. 4 we showed that $\mathscr{L}$ is u_0-positive with respect to the cone of positive functions on $D_{\mathbb{R}}=D\cap\mathbb{R}$ and derived in this way a new proof of Kuzmin's theorem for the Gauss map and other piecewise expanding maps also in higher dimensions. For the present discussion a slightly different space of holomorphic functions appears to be more suited. Denote by $A_{1,\infty}(D)$ the Banach space of holomorphic functions over D, which together with their first derivatives are continuous on $\bar{D}$, together with the sup norm

$$\|f\|=\max\{\sup_{z\in\bar{D}}|f(z)|, \sup_{z\in\bar{D}}|f'(z)|\} \tag{2.2}$$

Trivially this space is a subspace of the space $A_\infty(D)$. The transfer operator $\mathscr{L}$ is a bounded linear operator on this space and it is straightforward to extend the above properties of $\mathscr{L}$ from the space $A_\infty(D)$ to the space $A_{1,\infty}(D)$: $\mathscr{L}$ can be decomposed as $\mathscr{L}=P_1+\mathscr{N}$, where P_1 is the projector onto the eigenfunction $h_1(z)=1/(z+1)\log 2$ corresponding to the eigenvalue $\lambda_1=1$. Its explicit form is given as

$$P_1 f(z)=h_1(z)\int_0^1 f(x)\,dx \tag{2.3}$$

$\mathscr{N}$ is some bounded linear operator with $P_1\mathscr{N}=\mathscr{N}P_1=0$ and spectral radius strictly smaller than unity. If $C_{f,g}(n)$ denotes the correlation function

$$C_{f,g}(n)=\int_0^1 d\mu(x)\,f(T^n x)\,g(x)-\int_0^1 d\mu(x)\,f(x)\int_0^1 d\mu(x)\,g(x)$$

then this function can be written for observables $f,\ g\in A_{1,\infty}(D)$ as

$$C_{f,g}(n)=\int_0^1 dx\,f(x)(\mathscr{L}-P_1)^n\,(h_1\cdot g)(x) \tag{2.4}$$

because $d\mu(x)=h_1(x)\,dx$. Kuzmin's theorem then follows from

$$|C_{f,g}(n)|\leqslant\rho^n\,\|g\|\,\|f\| \qquad \text{for}\quad f,\ g\in A_{1,\infty}(D)$$

where ρ denotes the spectral radius of $\mathscr{N}$. From this we see that the relaxation time τ for T is simply given as $\tau=-1/\log|\lambda_2|$, where λ_2 is in fact

[2] A brief review of this theory can be found in Ref. 8, Appendix A.

the dominant eigenvalue of $\mathcal{N}$ in absolute value determining the spectral radius of $\mathcal{N}$, as we will show. In Ref. 3 we also proved simple trace formulas for the operators $\mathcal{L}^n$, relating them to the fixed points of the map T^n:

$$\text{trace } \mathcal{L}^n = \sum_{i_1,\dots,i_n=1}^{\infty} \left[\prod_{k=1}^{n} (x^2_{i_k, i_{k+1},\dots, i_n, i_1,\dots, i_{k-1}})^{-1} - (-1)^n \right]^{-1} \tag{2.5}$$

valid also in the space $A_{1,\infty}(D)$, where $x_{i_1,\dots,i_n}$ denotes the number $x \in [0, 1]$ whose continued-fraction expansion is periodic of period n with entries $i_1,\dots, i_n$. It is obvious that these numbers are just the fixed points of the map T^n.

Before discussing the eigenvalue λ_2 in some detail, we first investigate the relation between the spectrum $\sigma(\mathcal{L})$ of the transfer operator $\mathcal{L}$ in the space $A_{1,\infty}(D)$ and the spectrum $\sigma(\hat{K})$ of the operator $\hat{K}$ in the Hilbert space $\mathcal{H}^2(v)$ as studied in I.

3. THE RELATION BETWEEN $\sigma(\mathcal{L})$ AND $\sigma(\hat{K})$

We know already that both spectra consist of eigenvalues accumulating at the point $\lambda = 0$. We want to show that indeed $\sigma(\hat{K}) = \sigma(\mathcal{L})$, and that $\mathcal{L}$ and $\hat{K}$ have exactly the same eigenvalues, multiplicities included. To show this, we start with the identity[2]

$$\mathcal{L}(h_1 \cdot f) = h_1 \cdot Vf \tag{3.1}$$

valid for all $f \in A_{1,\infty}(D)$, where $V: A_{1,\infty}(D) \to A_{1,\infty}(D)$ denotes the operator

$$Vf(z) = \sum_{n=1}^{\infty} \frac{z+1}{(z+n)(z+n+1)} f\left(\frac{1}{z+n}\right) \tag{3.2}$$

V therefore has exactly the same form as the operator $\hat{K}$ in I, the only difference being the different spaces on which they act. From (3.1) it is clear that $\sigma(\mathcal{L}) = \sigma(V)$ and all eigenvalues have the same algebraic multiplicities. This shows that for arbitrary n

$$\text{trace } \mathcal{L}^n = \text{trace } V^n \tag{3.3}$$

because V is also of trace class in the above space.

We now show the following result:

Theorem 1. The spectra $\sigma(\mathcal{L})$ and $\sigma(\hat{K})$ are identical, algebraic multiplicities of the eigenvalues included.

Proof. Take any eigenvalue $\lambda \in \sigma(\hat{K})$. Then we know already from I that $\lambda \neq 0$. Denote by $\hat{\varphi}$ the corresponding eigenfunction in $\mathscr{H}^2(\nu)$ such that $\hat{K}\hat{\varphi} = \lambda\hat{\varphi}$. From Theorem 5 in I we know that

$$\hat{\varphi}(z) = (z+1)\int_0^\infty dm(t)\, e^{-zt}\varphi(t) \tag{3.4}$$

where $\varphi \in L_2(m)$ satisfies the relation

$$\lambda\varphi(t) = \int_0^\infty dm(s)\, J_1(2(st)^{1/2})/(st)^{1/2}\, \varphi(s)$$

and $dm(t) = t\, dt(e^t - 1)^{-1}$.

This shows that φ is bounded on $[0, \infty)$. But from relation (3.4) we see that $\hat{\varphi}$ is holomorphic in every half-plane $\operatorname{Re} z > -1 + \delta$, $\delta > 0$. This shows that $\hat{\varphi} \in A_{1,\infty}(D)$ and that therefore $\sigma(\hat{K}) \subset \sigma(\mathscr{L})$. Take, on the other hand, any $\lambda \in \sigma(\mathscr{L})$. We will show first that $\lambda \neq 0$. Otherwise there exists a function $0 \neq f \in A_{1,\infty}(D)$ such that

$$0 = \sum_{n=1}^{\infty} \frac{1}{(z+n)^2} f\left(\frac{1}{z+n}\right)$$

If $g(z)$ denotes the function $f(1/(z+1))$, we see that $g(z)$ is holomorphic in the domains $\operatorname{Re} z > -1/2$ and $|z+1| > 2$. Furthermore, we have that $|g(z)| < M$ in the last domain for some $M > 0$. The eigenvalue equation for $g(z)$ reads

$$g(z) = -(z+1)^2 \sum_{n=2}^{\infty} \frac{1}{(z+n)^2}\, g(z+n-1) \tag{3.5}$$

From this equation one derives recursively that $g(z)$ can be analytically continued to the domain $\operatorname{Re} z > -7/2$ and defines in this way a holomorphic function in the whole complex plane. Because, as we saw, g is bounded in the domain $|z+1| > 2$, it is bounded everywhere and therefore a constant. Relation (3.5) then shows that g must vanish identically and therefore also f. But then $\lambda \neq 0$.

Let $f \in A_{1,\infty}(D)$ be the eigenfunction corresponding to the eigenvalue $0 \neq \lambda \in \sigma(\mathscr{L}) = \sigma(V)$ such that

$$\lambda f(z) = (z+1) \sum_{n=1}^{\infty} \frac{1}{(z+n)(z+n+1)} f\left(\frac{1}{z+n}\right) \tag{3.6}$$

From this one derives again recursively that f is holomorphic in every half-plane $H_\delta = \{z \colon \operatorname{Re} z > 1 - \delta\}$ for all $\delta > 0$. There one uses the contraction properties of the maps $\psi_n(z)=1/(z+n)$.[4] For any of these half-planes there exists a bounded domain B_δ in D such that $\psi_n(H_\delta)\subset B_\delta$ for all $n\geqslant 1$. Hence there exists $M_\delta>0$ such that

$$\sup_{z\in H_\delta, n\geqslant 1} |f(\psi_n(z))|\leqslant M_\delta \tag{3.7}$$

This shows that f in (3.6) is bounded in every H_δ for $\delta>0$. We still have to show that $\int d\nu(z)\,|f(z)|^2<\infty$, where ν denotes the measure

$$d\nu(z)=\frac{1}{\pi}\frac{dx\,dy}{(1+x)^2+y^2}$$

in the strip $-1/2\leqslant x\leqslant 0$.

Inserting this expression and using the eigenvalue equation (3.6), we find

$$\begin{aligned}
&\int d\nu(z)\,|f(z)|^2\\
&\quad=\frac{1}{\lambda^2\pi}\int_{-1/2}^{0}dx\int_0^\infty dy\left|\sum_{n=1}^\infty\frac{1}{(z+n)(z+n+1)}f\left(\frac{1}{z+n}\right)\right|^2\\
&\quad\leqslant C\sum_{n=1}^\infty\sum_{m=1}^\infty\int_{-1/2}^{0}dx\int_0^\infty dy\frac{1}{(x+n)^2+y^2}\frac{1}{(x+m)^2+y^2}
\end{aligned}$$

for some constant $C>0$. Schwartz's inequality then shows that

$$\int d\nu(z)\,|f(z)|^2\leqslant C'\left[\sum_{n=1}^\infty\left\{\int_0^\infty dy\frac{1}{[(n-1/2)^2+y^2]^2}\right\}^{1/2}\right]^2$$

Because

$$\int_0^\infty dy\frac{1}{[(n-1/2)^2+y^2]^2}=\frac{1}{(n-1/2)^3}\frac{\Gamma(1/2)\,\Gamma(3/2)}{2\Gamma(2)}$$

we finally get

$$\int d\nu(z)\,|f(z)|^2\leqslant C''\left(\sum_{n=1}^\infty\frac{1}{(n-1/2)^{3/2}}\right)^2<\infty$$

and therefore $f\in\mathscr{H}^2(\nu)$. This, however, shows that $\sigma(\mathscr{L})\subset\sigma(\hat{K})$ and therefore $\sigma(\mathscr{L})=\sigma(\hat{K})$.

Let us finally discuss the algebraic multiplicities of the different eigenvalues. Because $\hat{K}$ is isomorphic to a self-adjoint operator K (see I), the algebraic multiplicity of any $\lambda \in \sigma(\hat{K})$ is equal to its geometric multiplicity. From our foregoing discussion of the eigenfunctions it follows that the algebraic multiplicity $m_{\mathscr{L}}(\lambda)$ of any eigenvalue $\lambda \in \sigma(\mathscr{L})$ fulfills

$$m_{\hat{K}}(\lambda) \leqslant m_{\mathscr{L}}(\lambda) \tag{3.8}$$

Theorem 3 in I and relation (2.5) for $n = 2$ show, however, that

$$\sum_i \left[m_{\hat{K}}(\lambda_i) - m_{\mathscr{L}}(\lambda_i) \right] \lambda_i^2 = 0 \tag{3.9}$$

Because all λ_i are real and different from zero, we have $\lambda_i^2 > 0$ and therefore because of relation (3.8) we must have

$$m_{\hat{K}}(\lambda) = m_{\mathscr{L}}(\lambda) = m_V(\lambda)$$

This concludes the proof of Theorem 1.

As a result, we get the trace formulas announced at the end of Section 3 in I for general n: the traces of the operators $\hat{K}^n$ (respectively K^n) can be expressed as in the case $n = 1$ and $n = 2$ by the fixed points $x_{i_1,\dots,i_n}$ of the map T^n:

$$\text{trace } K^n = \sum_{i_1,\dots,i_n = 1}^{\infty} \left[\prod_{k=1}^{n} x_{i_k,\dots,i_n,i_1,\dots,i_{k-1}}^{-2} - (-1)^n \right]^{-1}$$

Hence Theorems 2 and 3 in I are special cases of this general result.

Let us add some remarks. We have not succeeded in proving reality of $\sigma(\mathscr{L})$ in $A_{1,\infty}(D)$ directly without using the relation $\sigma(\mathscr{L}) = \sigma(\hat{K})$ of Theorem 1. It would be interesting to understand this property within the Banach space approach from the special form of the transfer operator. We come back to this question at the end of this paper. The only case we understand is the eigenvalue λ_2, which we discuss next.

4. u_0-POSITIVITY OF THE OPERATOR $\mathscr{N}$

We want to apply the theory of composition operators developed in Refs. 9 and 10, which is based on Krasnoselskii's[11] work on u_0-positive operators in general Banach spaces. We used this technique in Ref. 4 to discuss Kuzmin's theorem for quite general locally expanding maps in $\mathbb{R}^k$. Our main concern there was the highest eigenvalue of the corresponding transfer operators. Here we are interested in their subdominant eigenvalue λ_2. In an earlier discussion of transfer operators for one-dimensional lattice spin systems[8] we found that subdominant eigenvalues also can be studied by this technique. Because our discussion is closely related to the one there, we can proceed rather rapidly.

To start with, let $A^{\mathbb{R}}_{1,\infty}(D)$ be the real subspace of $A_{1,\infty}(D)$ of all f that take real values on $D_{\mathbb{R}} = \mathbb{R} \cap D$. Because every eigenfunction h_i of $\mathscr{L}$ belonging to an eigenvalue $\lambda_i \neq \lambda_1$ satisfies[4]

$$\int_0^1 h_i(x)\, dx = 0 = P_1 h_i$$

every corresponding eigenfunction $\hat{h}_i$ of the operator V in (3.2) fulfills

$$\int_0^1 \hat{h}_i(x)\, h_1(x)\, dx = 0 \tag{4.1}$$

Hence, we can restrict our discussion to the subspace

$$\tilde{A}^{\perp}_{1,\infty}(D) = \left\{ f \in A_{1,\infty}(D) \colon \int_0^1 f(x)\, dx = 0 \right\}$$

We find because of the general relation[2]

$$\int_0^1 \mathscr{L} f(x)\, dx = \int_0^1 f(x)\, dx \tag{4.2}$$

that $\mathscr{L}$ leaves this subspace $\tilde{A}^{\perp}_{1,\infty}(D)$ invariant and

$$\mathscr{L} \mid_{\tilde{A}^{\perp}_{1,\infty}(D)} = \mathscr{N} \mid_{\tilde{A}^{\perp}_{1,\infty}(D)} \tag{4.3}$$

For $f \in \tilde{A}^{\perp}_{1,\infty}(D)$ of the form $f = \hat{f} \cdot h_1$ we get from relation (3.1)

$$\mathscr{N}(\hat{f} \cdot h_1) = h_1 \cdot Vf$$

so that the spectrum $\sigma(\mathscr{N})$ of $\mathscr{N}$ is identical to the spectrum $\sigma(V)$ of the operator V in the space

$$A^{\perp}_{1,\infty}(D) = \left\{ \hat{f} \in A_{1,\infty}(D) \colon \int_0^1 \hat{f}(x)\, h_1(x)\, dx = 0 \right\} \tag{4.4}$$

Consider next the real subspace $A^{\perp,\mathbb{R}}_{1,\infty}(D)$ of all $\hat{f} \in A^{\perp}_{1,\infty}(D)$ that take real values on $D_{\mathbb{R}}$, and in this space the cone C defined as

$$C = \{ \hat{f} \in A^{\perp,\mathbb{R}}_{1,\infty}(D) \colon \hat{f}'(x) \geqslant 0 \text{ on } D_{\mathbb{R}} \} \tag{4.5}$$

Because from $\hat{f}$ and $-\hat{f} \in C$ it follows that $\hat{f}' \equiv 0$ on $D_{\mathbb{R}}$ and therefore $\hat{f} \equiv \text{const}$ on D, the property $\int_0^1 \hat{f}(x)\, h_1(x)\, dx = 0$ shows that $\hat{f} \equiv 0$. Hence C is a proper cone.

To show that C is reproducing, take any $\hat{f} \in A_{1,\infty}^{\perp,\mathbb{R}}(D)$. Since $\hat{f}'$ is continuous on $\bar{D}$,

$$M = \max_{x \in \bar{D}_{\mathbb{R}}} |\hat{f}'(x)|$$

exists and is finite. The function $\hat{u}_0(z) = z + 1 - 1/\log 2$ is obviously an element of C, as is the function $\hat{g}_1(z) = M\hat{u}_0(z)$. Then the function $\hat{g}_2(z) = \hat{f}(z) + \hat{g}_1(z)$ fulfills $\hat{g}_2'(x) = \hat{f}'(x) + \hat{g}_1'(x) \geqslant 0$ and therefore $\hat{g}_2 \in C$. Hence, we can write $\hat{f}(z) = \hat{g}_2(z) - \hat{g}_1(z)$ with both $\hat{g}_i \in C$. This shows that C is reproducing. A final property of C we should mention is that $\mathring{C} =$ int $C \neq \varnothing$, since, for instance, $\hat{u}_0(z)$ is such an interior element.

We can then show the following result:

Lemma 1. The operator $-V: A_{1,\infty}^{\perp,\mathbb{R}}(D) \to A_{1,\infty}^{\perp,\mathbb{R}}(D)$ is positive with respect to the cone C.

Proof. Let $\hat{f}$ be an arbitrary element in C. Then we have

$$(-V\hat{f})(z) = -\sum_{n=1}^{\infty} a_n(z)\, \hat{f}\left(\frac{1}{z+n}\right) \tag{4.6}$$

where $a_n(z)$ are holomorphic functions in D given by

$$a_n(z) = \frac{z+1}{(z+n)(z+n+1)}$$

Because $\sum_{n=1}^{\infty} a_n(z) = 1$ uniformly in $\bar{D}$, we have in the same domain

$$\sum_{n=1}^{\infty} a_n'(z) = 0 \tag{4.7}$$

Differentiating expression (4.6) gives

$$(-V\hat{f})'(z) = -\sum_{n=1}^{\infty} a_n'(z)\, \hat{f}\left(\frac{1}{z+n}\right) + \sum_{n=1}^{\infty} a_n(z) \frac{1}{(z+n)^2} \hat{f}'\left(\frac{1}{z+n}\right) \tag{4.8}$$

Restricting the argument to $D_{\mathbb{R}}$, we see that the terms in the second sum are all nonnegative, so that this sum is nonnegative. To discuss the first sum, we deduce from relation (4.7) that we can assume $\hat{f}$ positive on $D_{\mathbb{R}}$: otherwise replace $\hat{f}$ by $\hat{f} + M$ and M large enough. Because $\hat{f}'(x) \geqslant 0$ on $D_{\mathbb{R}}$, the function $\hat{f}(1/(x+n))$ is monotone decreasing in n. Consider next the functions $-a_n'(x)$. Their explicit form is

$$-a_n'(x) = \frac{1}{(x+n)(x+n+1)} \left(\frac{x+1}{x+n} + \frac{x+1}{x+n+1} - 1\right) \tag{4.9}$$

This shows that there exists for every $x \in \bar{D}_{\mathbb{R}}$ an integer $m = m(x)$ such that

$$-a_n'(x) \begin{cases} \geqslant 0 & \text{for} \quad n \leqslant m(x) \\ < 0 & \text{for} \quad n > m(x) \end{cases}$$

But then

$$-\sum_{n=1}^{m(x)} a_n'(x)\, \hat{f}\left(\frac{1}{x+n}\right) \geqslant \hat{f}\left(\frac{1}{x+m(x)}\right) \sum_{n=1}^{m(x)} -a_n'(x) \geqslant 0$$

On the other hand, relation (4.7) gives

$$\sum_{n=1}^{m(x)} -a_n'(x) = \sum_{n=m(x)+1}^{\infty} a_n'(x)$$

and therefore

$$\hat{f}\left(\frac{1}{x+m(x)}\right) \sum_{n=1}^{m(x)} -a_n'(x) = \hat{f}\left(\frac{1}{x+m(x)}\right) \sum_{n=m(x)+1}^{\infty} a_n'(x)$$
$$\geqslant \sum_{n=m(x)+1} a_n'(x)\, \hat{f}\left(\frac{1}{x+n}\right) \geqslant 0$$

This shows that

$$-\sum_{n=1}^{\infty} a_n'(x)\, \hat{f}\left(\frac{1}{x+n}\right) \geqslant 0$$

and therefore $(-V\hat{f})(x) \geqslant 0$ on $\bar{D}_{\mathbb{R}}$ for $\hat{f} \in C$.

For the following discussion let us briefly recall the notion of a u_0-positive operator P in a real Banach space B with cone K[11]:

Definition. For $u_0 \in K$, $u_0 \neq 0$, the positive operator $P\colon B \to B$ is called u_0-positive if there exist for every $f \in K$, $f \neq 0$, constants $\alpha = \alpha(f) > 0$, $\beta = \beta(f) > 0$, and an integer $n = n(f)$ such that $\alpha u_0 \leqslant_K P^n f \leqslant_K \beta u_0$, where $\leqslant_K$ denotes the ordering induced by the cone K on B.

As before, we denote by $\hat{u}_0$ the function $\hat{u}_0(z) = z + 1 - 1/\log 2$ in the cone C defined in (4.5). Then we have the following result:

Lemma 2. The operator $-V\colon A_{1,\infty}^{+,\mathbb{R}}(D) \to A_{1,\infty}^{+,\mathbb{R}}(D)$ is u_0-positive.

Proof. Take any $\hat{f} \in C$, $\hat{f} \neq 0$. We know already that $-V\hat{f} \in C$. Because $(-V\hat{f})'$ is continuous on $\bar{D}_{\mathbb{R}}$, both

$$\alpha = \max_{x \in \bar{D}_{\mathbb{R}}} (-V\hat{f})'(x) \qquad \text{and} \qquad \beta = \min_{x \in \bar{D}_{\mathbb{R}}} (-V\hat{f})'(x)$$

exist and are nonnegative. Assume $\beta=0$. Then there exists $\bar{x}\in\bar{D}_{\mathbb{R}}$ with $(-V\hat{f})'(\bar{x})=0$. Inserting expression (4.8) and using the fact that both sums are nonnegative, this being true in the second sum for every term, we get, for all $n\in\mathbb{N}$,

$$\hat{f}\left(\frac{1}{\bar{x}+n}\right)=0$$

Because the set $\{1/(\bar{x}+n): n\in\mathbb{N}\}$ is a set of uniqueness for the holomorphic function $\hat{f}'$, we get $\hat{f}'\equiv 0$ and hence $\hat{f}\equiv\text{const}$ on $D_{\mathbb{R}}$. Because, however, $\hat{f}\in A_{1,\infty}^{\perp,\mathbb{R}}(D)$, we see that $\hat{f}\equiv 0$. Therefore β must be positive and

$$\beta\hat{u}_0 \leqslant_C -V\hat{f} \leqslant_C \alpha u_0 \tag{4.10}$$

An immediate consequence of Lemma 2 is the following result:

Theorem 2. The operator $-\mathcal{N}: \tilde{A}_{1,\infty}^{\perp}(D)\to\tilde{A}_{1,\infty}^{\perp}(D)$ has a simple positive dominant eigenvalue $-\lambda_2$ with eigenfunction $h_2=\hat{h}_2\cdot h_1$, where $\hat{h}_2\in\mathring{C}$. There is no further eigenfunction in $h_1\cdot C$.

Proof. Because $A_{1,\infty}^{\perp}(D)$ is just the complexified Banach space $A_{1,\infty}^{\perp,\mathbb{R}}(D)$, the theorem follows from Lemma 2 and Krasnoselskii's theorem on u_0-positive operators in Ref. 8, Appendix C.

Corollary. The operator $\mathscr{L}: A_{1,\infty}(D)\to A_{1,\infty}(D)$ has a simple positive dominant eigenvalue $\lambda_1=1$ and a simple negative subdominant eigenvalue λ_2 with $|\lambda_i|<-\lambda_2<\lambda_1=1$ for all $i\geqslant 3$.

The problem now is to determine the eigenvalue λ_2. But this can be achieved by a minimax principle, which follows immediately from the foregoing result as derived in a similar situation in Ref. 8:

$$\min_{\hat{f}\in\mathring{C}}\max_{x\in\bar{D}_{\mathbb{R}}}\frac{(V\hat{f})'(x)}{\hat{f}'(x)}=\lambda_2=\max_{\hat{f}\in\mathring{C}}\min_{x\in\bar{D}_{\mathbb{R}}}\frac{(V\hat{f})'(x)}{\hat{f}'(x)} \tag{4.11}$$

From this we get rigorous upper and lower bounds for the eigenvalue λ_2:

$$\min_{x\in\bar{D}_{\mathbb{R}}}\frac{(V\hat{f})'(x)}{\hat{f}'(x)}\leqslant\lambda_2\leqslant\max_{x\in\bar{D}_{\mathbb{R}}}\frac{(V\hat{f})'(x)}{\hat{f}'(x)} \tag{4.12}$$

where $\hat{f}$ is any element in the interior $\mathring{C}$ of the cone C. The above minimax principle has to be compared with a completely analogous one for the highest eigenvalue λ_1 of the operator $\mathscr{L}$:

$$\max_{f\in\mathring{K}_+}\min_{x\in\bar{D}_{\mathbb{R}}}\frac{\mathscr{L}f(x)}{f(x)}=\lambda_1=\min_{f\in\mathring{K}_+}\max_{x\in\bar{D}_{\mathbb{R}}}\frac{\mathscr{L}f(x)}{f(x)}$$

where K_+ is the cone

$$K_+ = \{f \in A^{\mathbb{R}}_{1,\infty}(D): f(x) \geqslant 0 \text{ on } \bar{D}_{\mathbb{R}}\}$$

in the real Banach space $A^{\mathbb{R}}_{1,\infty}(D)$. In this case, however, the formula is not very interesting, because by relation (4.2), λ_1 must be 1 anyhow.

Remark. By using, for instance, the test function $f(z) = (z+1)/(z+r) - c$ with $r = 1.14617$ and c chosen such that $f \in A^{\perp}_{1,\infty}(D)$, we get the following numerical bounds on λ_2:

$$-0.2995 \geqslant \lambda_2 \geqslant -0.3038$$

which has to be compared with the bounds in I, which were obtained by a more sophisticated procedure. Taking more general test functions with more than one free parameter will certainly improve the above bounds considerably.

Let us close with some conjectures concerning the spectral properties of a class of composition operators on spaces of holomorphic functions of the form we found for the transfer operator for the Gauss map. In Refs. 9 and 10 we discussed composition operators of the form

$$\mathscr{L}f(z) = f \circ \psi(z)$$

on the space $A_\infty(D)$, $D \subset \mathbb{C}^n$, where $\psi: D \to D$ is a holomorphic map of D into itself, which furthermore leaves $D_{\mathbb{R}^n} = D \cap \mathbb{R}^n$ invariant and has its unique fixed point in $D_{\mathbb{R}^n}$. We found that $\mathscr{L}$ has only real, simple eigenvalues under these conditions. The results on the transfer operator $\mathscr{L}$ for the Gauss map (respectively other piecewise linear expanding maps of the unit interval) suggest that the following conjecture is true:

Conjecture 1. Let $\psi_i: D \to D$ be a finite or countable family of holomorphic maps of a domain $D \subset \mathbb{C}$ into itself such that $\psi_i: D_{\mathbb{R}} \to D_{\mathbb{R}}$. Assume that all the fixed points of the maps ψ_i belong to $D_{\mathbb{R}}$. Then the spectrum of the linear operator

$$\mathscr{L}f(z) = \sum_i \varphi_i f \circ \psi_i(z)$$

where φ_i are holomorphic on D and real on $D_{\mathbb{R}}$, is real and every eigenvalue is simple.

In case of the transfer operator $\mathscr{L}$ for the Gauss map we expect the following conjecture to be true:

Conjecture 2. If $\psi_i(z) = 1/(z+i)$ and $\varphi_i(z) = 1/(z+i)^2$, which means $\mathscr{L}$ is the transfer operator for the Gauss map T, the eigenvalues λ_i alternate in sign, that is, $(-1)^{i+1} \lambda_i > 0$.

One could think that Conjecture 1 is related to the existence of a self-adjoint operator in some Hilbert space which has the same spectrum as $\mathscr{L}$, as we found in the case of the Gauss map. Work in this direction is in progress and we hope to come back to these conjectures soon.

An interesting application of Conjecture 1 would be that Ruelle's[12] zeta function for certain expanding systems has only simple poles, which furthermore all lie on the real axis. For the Gauss map this follows from our results in this and the former paper.[1]

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